Small Area Confidence Bounds on Small Cell Proportions in Survey Populations

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Outline

- General Problem
- Cell-Based vs. Model-Based Methods
- Alternative Definitions of *Effective Sample Size*
- Data Example – Erroneous Enumeration of HUs in Census Coverage Measurement study (CCM)
- Numerical Results
- Conclusion and Future Plans

General Problem

Setting: given direct (ratio) estimates for small proportions $\widehat{\pi}_i$ at level of "cell" or domain i (e.g., county)

Problem: specify upper confidence bound for $\widehat{\pi}_i$

- `either` bound in transformed measurement scale $h(\pi_i)$ from data within domain i
- `or` model-based bound connecting values π_i across domains using predictors $\mathbf{x}_i$

Approach: using *effective* sample-sizes n_i^*, adapt binomial/SRS estimator $\widehat{Var}(\widehat{\pi}_i) = \widehat{\pi}_i\,(1-\widehat{\pi}_i)/n_i^*$

Applications

- demographic tables in **ACS**,

  the American Community Survey;

- area-level Erroneous Enumeration rates in

  CCM, Census Coverage Measurement;

- and rates in other Census Bureau surveys.

A Good Cell-Based Method

`Liu and Kott 2009, Survey Methodology`

π_i = true proportion

n_i^* = effective sample size, $y_i^* \sim$ Binom(n_i^*, π_i)

$$\widehat{\pi}_i = \frac{y_i}{n_i} = \frac{y_i^*}{n_i^*} \qquad \textit{direct estimator}$$

Transformation: $asin(\sqrt{\widehat{\pi}})$ `(Variance-stabilizing)`

centered at $asin(\sqrt{\pi})$, Var $\approx 1/(4n_i^*)$

Obtain UCB on arcsin scale,

transform back to prob. scale by $sin^2(x)$

Ideas of Model-Based Methods

π_i includes $\begin{cases} \text{predicted part} & \eta_i = \mathbf{x}_i' \beta \\ \text{unmodeled random component} \end{cases}$

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(1) Fay-Herriot:
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$$\pi_i = sin^2(\eta_i + u_i), u_i \sim N(0, \sigma_u^2)$$

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(2) Logistic:
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$$\pi_i = \frac{e^{\eta_i + v_i}}{1 + e^{\eta_i + v_i}}, v_i \sim N(0, \sigma_v^2)$$

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(3) Beta-Binomial:
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$$\pi_i = \text{Beta}\left(\frac{\tau\, e^{\eta_i}}{1 + e^{\eta_i}}, \frac{\tau}{1 + e^{\eta_i}} \right)$$

Parameters σ_u^2 , σ_v^2 , $(1 + \tau)^{-1}$ quantify *imprecision* of π_i in terms of η_i

Model-Based Methods, Continued

Recall $\quad y_i^*/n_i^* \; = \; \widehat{\pi}_i$

`Fay-Herriot:`

$$\text{arcsin}(\sqrt{\widehat{\pi}_i}) \;\sim\; \mathbf{N}(\text{arcsin}(\sqrt{\pi}_i),\, \frac{1}{4n_i^*})$$

`Logistic & Beta-Binomial:`

$$y_i^* \;\sim\; Binom(n_i^*, \pi_i)$$

Estimation in Model-Based Methods

Point Predictor for π_i

generally different from direct estimator

BLUP: $E(\pi_i \mid y_i^*)$

EBLUP: substitute MLE for β & (σ_u, σ_v or τ)

Upper Confidence Bound for π_i

based on $\widehat{Var}(EBLUP)$

Defining Effective Sample Size n_i^*

If variance $\frac{V_i}{n_i} = \frac{\pi_i(1-\pi_i)}{n_i^*}$ of direct π_i estimator in area i is reliably estimated and sampling fraction $f \ll 1$:

$$\text{DEFF}_i = \frac{V_i}{\pi_i\,(1-\pi_i)} \quad , \qquad n_i^* = \frac{n_i}{\text{DEFF}_i}$$

What if $\widehat{V}_i$ is erratic or π_i too small ?

`Proposal 1:` Area size via higher-level DEFF

With $\widehat{V}/n = \widehat{V}(\widehat{\pi})$, and DEFF at higher (e.g., State) level

$$\text{DEFF} = \frac{\widehat{V}}{\pi(1-\pi)} \quad , \qquad n_i^* = \frac{n_i}{\text{DEFF}}$$

Effective Sample Size, Continued

`Proposal 2:` Eff. size modified by sampling weights

w_{ik} indiv.-level sampling weights within area i

$$n_i^* = \frac{n_i}{\text{DEFF}} \cdot \left(\frac{\sum_k w_{ik}^2}{(\sum_k w_{ik})^2} \right) \Big/ \left(\frac{\sum_{j,k} w_{jk}^2}{(\sum_{j,k} w_{jk})^2} \right)$$

Effective sizes for survey CIs : Liu & Kott 2009

in Bayesian analysis: Chen et al. 2011, Malec 2005

Data Example

- Census Coverage Measurement (CCM):
  - evaluation of Census performance.
  - finds Erroneous Enumeration (EE) rates for counties in sample.
  - 170k Housing Units (HUs) and 1,728 counties.
  - Census publishes detailed estimates for 128 counties with pop. $\geq$ 500k

Specific CCM Details

- National EE rate $\approx$ 2.7% for metro areas (Olson, 2012).
- For our 128 counties, P($\widehat{\pi}_i = 0$) $>$ 0.

EE Rates (*i* = county, *k* = HU):

$$\widehat{\pi} = \frac{\Sigma_{j,k} W_{jk} Y_{jk}}{\Sigma_{j,k} W_{jk}} \quad , \qquad \widehat{\pi}_i = \frac{\Sigma_k W_{ik} Y_{ik}}{\Sigma_k W_{ik}}$$

Specific CCM Modeling Details

- Area- or cell-level models, all covariates from Census.

BIC model-selection penalizes max logLik by $k\,\ln(n)$

($k =$ # regr. coef's, $n =$ total sample size).

Our selected model had 5 predictor variables:

- state EE rate, a synthetic estimator;
- area rate of single-unit households;
- area rate of large multi-unit households;
- area rate of urban households;
- area enumeration rate.

Variability Across Counties

n^* = Proposal 2 Eff. S.S., $n^\dagger$ = Proposal 1 Eff. S.S.

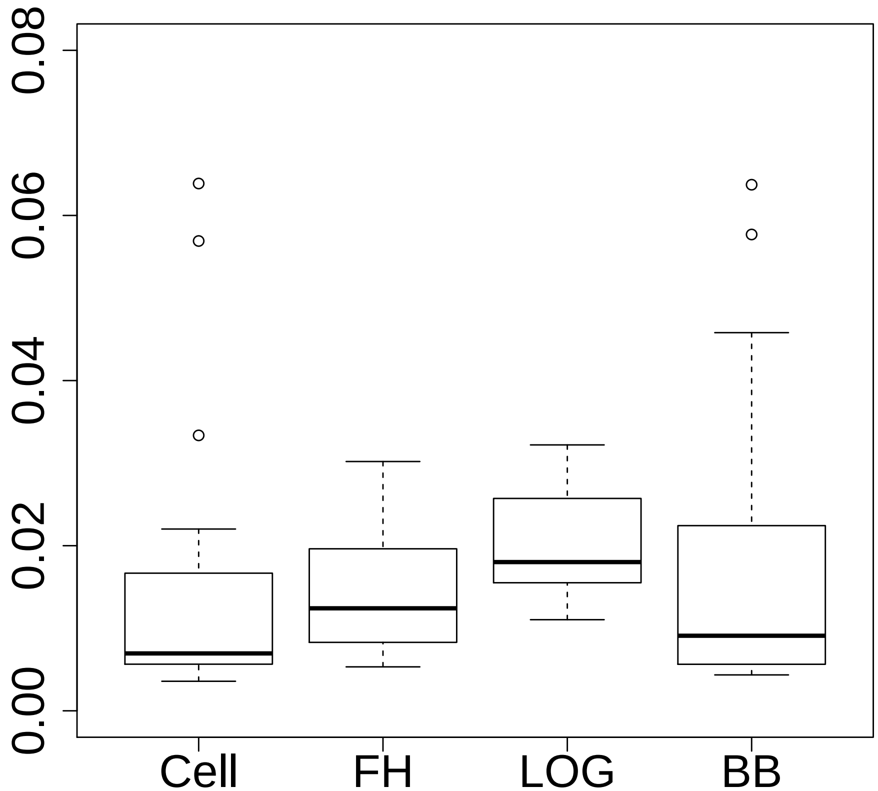
		Mean	**1st Q**	**Med.**	**3rd Q**
`St`	DEFF	6.8	2.7	3.8	6.9
`Cou`	n^*	34.6	8.5	15.8	34.2
`Cou`	$n^*/n^\dagger$	1.2	1.1	1.1	1.2

NB: n^*, $n^\dagger$ based on counties with $\geq$ 20 HUs in sample.

Results

- Inclusion of n^* led to wider, more conservative estimates for the UCBs.
- The Fay-Herriot and Logistic methods had slightly higher means and medians for UCBs for areas with $\widehat{\pi} \approx 0$, but the Cell and Beta-Binomial methods had the highest maximum UCBs.
- Overall, all four methods created upper bounds within a similar range.

UCB of Production Counties where $\widehat{\pi}_i$ = 0



Conclusion & Future Plans

Conclusions:

• For our project, we chose the Cell-Based method. With the results so close, a simpler method without a specified regression model was preferrable.

• We used the Proposal 2 n_i^* for accurately capturing the variance because it did not assume SRS or equal weights within area. Results were more conservative which was a priority in this case.

Future plans:

• extend approach to other applications

• test performance of DEFF under different assumptions?

• generalized R package for Census Bureau use?

References

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Thank You!

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A Cell-Based Method

Variance-Stabilizing transformation to *arcsin sqrt* scale
of estimated area proportion $y_i/n_i \;=\; y_i^*/n_i^*$

For area *i* : π_i = true proportion, n_i^* = eff. samp. size

$$y_i^* \;\sim\; Binom(n_i^*, \pi_i) \;\approx\; \mathcal{N}(n_i^*\,\pi_i,\, n_i^*\,\pi_i\,(1-\pi_i))$$

$$\text{arcsin}\,\sqrt{\frac{y_i^*}{n_i^*}} \;\approx\; \mathcal{N}(\text{arcsin}\,\sqrt{\pi_i},\; \frac{1}{4n_i^*}\,) \qquad \Delta\textit{-method}$$

Transformed scale 90% `CI:` $\text{arcsin}\,\sqrt{y_i^*/n_i^*} \pm 1.645/\sqrt{4\,n_i^*}$

Transform back to the probability scale by $\sin^2(x)$

Models which Borrow Strength across Areas

`Notation:` $\widehat{\pi}_i = \frac{y_i}{n_i} = \frac{y_i^*}{n_i^*}$, $\widehat{a}_i = \arcsin\sqrt{\widehat{\pi}_i}$, $\eta_i = \mathbf{x}_i'\,\beta$

$\mathbf{x}_i$ `area-level observed predictors`

$u_i \sim \mathcal{N}(0, \sigma_u^2)$, $v_i \sim \mathcal{N}(0, \sigma_v^2)$ `random effects`

(1) Fay-Herriot : $\widehat{a}_i = \eta_i + u_i + \epsilon_i$, $\epsilon_i \sim \mathcal{N}(0, \frac{1}{4n_i^*})$

(2) Logistic Random-Intercept : $y_i^* \sim \mathrm{Bin}(n_i^*, \frac{e^{\eta_i+v_i}}{1+e^{\eta_i+v_i}})$

(3) Beta-Binomial : $y_i^* \sim \mathrm{Bin}(n_i^*, \pi_i)$, $\pi_i \sim \mathrm{Beta}(\frac{\tau\, e^{\eta_i}}{1+e^{\eta_i}}, \frac{\tau}{1+e^{\eta_i}})$

Targets for small-area prediction: $\sin^2(\eta_i + u_i)$ in **(1)**; $\frac{e^{\eta_i+v_i}}{1+e^{\eta_i+v_i}}$ in **(2)**; and π_i in **(3)**.

Fay-Herriot Model UCB

Mod1 (FH): EBLUP $\widehat{\pi}_i = \sin^2(\widehat{\theta}_i/n_i^*)$, based on $\theta_i = \eta_i + \mu_i$,

$$\widehat{\theta}_i \;=\; \widehat{\gamma}_i\, y_i + (1 - \widehat{\gamma}_i)\, \mathbf{x}_i'\widehat{\beta} \quad , \quad \widehat{\gamma}_i = \frac{\widehat{\sigma}_u^2}{\widehat{\sigma}_u^2 + (4n_i)^{-1}}$$

$$UCB_i \;=\; \sin^2\left(\frac{1}{n_i^*}(\widehat{\theta}_i + \frac{z_\alpha}{2}((1-\widehat{\gamma}_i)\,\widehat{\sigma}_u^2 + (1-\widehat{\gamma}_i)^2 \mathbf{x}_i'\,\widehat{V}_{\widehat{\beta}}\,\mathbf{x}_i\}^{1/2})\right)$$

Fay and Herriot (1979), Slud (2012)

NB: includes sample variability of $\widehat{\beta}$, not $\widehat{\sigma}_u^2$

Rao (2003) has more inclusive formulas for related $\widehat{\text{mse}}$

Logistic Random-Intercept Model-Based UCB

Mod2 (LgstRI): EBLUP $\widehat{\pi}_i \;=\; \dfrac{g(y_i^* + 1,\, n_i^* + 1,\, \widehat{\eta}_i,\, \widehat{\omega}^2)}{g(y_i^*,\, n_i^*,\, \widehat{\eta},\, \widehat{\omega}^2)}$

where $\widehat{\eta}_i = \mathbf{x}_i' \,\widehat{\beta}$, $\widehat{\omega}^2 = \widehat{\sigma}_v^2 + \mathbf{x}_i' \,\widehat{V}_{\widehat{\beta}}\, \mathbf{x}_i$

$$g(k, n, \eta, \omega^2) \;=\; \int \frac{e^{(\eta+\omega z)k}}{(1 + e^{\eta+\omega z})^n}\, \phi(z) dz \;, \quad \phi(\cdot) \sim \mathcal{N}(0, 1)$$

$$UCB_i \;=\; \widehat{\pi}_i \;+\; 1.645 \left[\frac{g(y_i^* + 2,\, n_i^* + 2,\, \widehat{\eta},\, \widehat{\omega}^2)}{g(y_i^*,\, n_i^*,\, \widehat{\eta},\, \widehat{\omega}^2)} \;-\; \widehat{\pi}_i^2 \right]^{1/2}$$

NB: includes sample variability of $\widehat{\beta}$, not $\widehat{\sigma}_v^2$.

Jiang and Lahiri 2006, Slud 2012

Beta-Binomial Model-Based UCB

Mod3 (Beta-Bin): Estimation via Posterior , $\mu_i = \frac{e^{\eta_i}}{1 + e^{\eta_i}}$,

$$\pi_i \,|\, y_i^* \;\sim\; \text{Beta}(\tau\,\mu_i + y_i^*,\; \tau\,(1 - \mu_i) + n_i^* - y_i^*)$$

Empirical Bayes $\widehat{\pi}_i = \dfrac{y_i^* + \widehat{\tau}\,\widehat{\mu}_i}{n_i^* + \widehat{\tau}}$

- Bootstrap approach to UCB